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The significance of Fuzzy Integro-Differential Equations (Fides) stimulated by many researchers in recent years

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***Abstract*---**The present procedure relies upon converting nonlinear FVIDE into to nonlinear equations which tackle by using iterative approximation technique. Approximation results state the efficiency and more accurate of the present technique by compare these results by the Haar Wavelet Method (HWM) with the closed form solution. We present here a basic an essential profitable method for settling a class of Fuzzy Volterra integro-differential equations (FVIDE).

***Keywords*---**Approximation results, Nonlinear FVIDE, Fuzzy Volterra integro-differential equations, Coefficient matrix, Operational matrix.

Introduction

Nonlinear phenomena's have found many applications in several of mathematics, engineering, and physical. Usually, data about these problems involved is pervaded with uncertainty, which emerge in the experiment part and measurement process furthermore the initial values are determined. Classical mathematics can't adapt to this circumstance. So, it is an important to denote some of mathematical apparatus to approve this uncertainty [1-15]. More insights about fuzzy mathematics are shown in [16-18] to know more insights about fuzzy mathematics. The approximation solvability of fuzzy initial value problems (FIVPs) have discussed recently by many authors [19-25]; but, generally, investigation about FIDEs is scarce, especially, discussion on a nonlinear case. Therefore, we develop new approximation techniques for solving the FIVPs. The form of this paper sorted as; Properties of fuzzy integral equations are displayed in section 2. In section 3, the FVIDE are discussed. In section 4, Analysis of HWM then we utilize it for illustrating the approximation solution of FVIDE. In section 5, the present methods are actualized for illustrating two solved examples with accuracy and efficiency. Conclusion is demonstrated in section 6.

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Properties of Adjustment Equations

We give some documentation used in fuzzy analytics. We begin with characterizing the fuzzy number.
A fuzzy number is a fuzzy set $y: R^1 \to E^1 = [0,1]$ perform the accompanying documentations and assets.

1. y Is semi upper continuous on R.
2. The support $\overline{\{u \quad R | y(u) > 0\}}$ is a compact set,
3. y Is a fuzzy convex set $(y(\ u + (1 - \ )v) \geq \min\{y(u), y(v)\}$ $u, v \ R, \ [0,1])$,
4. There are real numbers a and $b, c \leq a \leq b \leq d,$ that
   a) $y(\alpha)$ is rising on $[c, a]$,
   b) $y(\alpha)$ is detraction on $[b, d]$,
   c) $y(\alpha) = 1$ for $a \leq \alpha \leq b.$

The fuzzy numbers set is defined by E^1 is denoted by Kaleva [13].

Generalized Construction Of the Vide

Here, we consider VIDE of the form;

$$y'(x) = f\big(x, y(x)\big) + \lambda \int_0^x k(x,t) y(t) dt \tag{1}$$

x

Where $\lambda > 0$, $k(t,s)$ kernel is function and $f \ (x, y(x)) + \lambda \int_0^x k(x,t) y(t) dt$ is a continuous fuzzy function. We assume that $\lambda > 0$. In order to design approximation scheme for solving Esq. (1), we write the parametric form of the given Esq. (1) as follows:

$$\underline{y}'(x,\alpha) = \underline{f}\left(x, \underline{y}(x,\alpha), \bar{y}(x,\alpha)\right) + \lambda \int_0^x \underline{U}(t,\alpha) dt$$

$$\bar{y}'(x,\alpha) = \bar{f}\left(x, \underline{y}(x,\alpha), \bar{y}(x,\alpha)\right) + \lambda \int_0^x \overline{U}(t,\alpha) dt \tag{2}$$

Where

$$\underline{U}(t,\alpha) = \underline{k(x,t) y(s,\alpha)} = \begin{cases} k(x,t) \ \underline{y}(t,\alpha), & k(x,t) \geq 0 \\ k(x,t) \ \bar{y}(t,\alpha), & k(x,t) < 0 \end{cases}$$

And

$$\overline{U}(t,\alpha) = \overline{k(x,t) y(s,\alpha)} = \begin{cases} k(x,t) \ \bar{y} \ (t,\alpha), & k(x,t) \geq 0 \\ k(x,t) \ \underline{y}(t,\alpha), & k(x,t) < 0 \end{cases} \tag{3}$$

Now we can say that Eq. (2) is a system of FVIDE for each $0 \leq \alpha \leq 1$ and $a \leq t \leq b$.

Haar Wavelet Method

The Haar wavelet functions is are an orthogonal function of switched rectangular waveforms where amplitudes can be different from one function to another and defined over the interval [0, 1] as follows

$$h_i(x) = \begin{cases} 1, & \varsigma < x < \beta \\ -1, & \beta < x < \gamma \\ o & elsewhere \end{cases}$$

(4)

Where $= \frac{k}{}$, $\beta = \frac{k+0.5}{m}$, and $\gamma = \frac{k+1}{m}$. The integer $m = 2^j \; where \; j = 0,1,2,\dots,J,$ denote the wavelet

level, and $k \overset{m}{=} 0,1,2,\dots,m-1$ is denote the translation parameter. Resolution level known as the integer
J. The index$i$ is established according to the formula $i = m + k + 1.$ In case of the values$m = 1,\; k =$
$0.$ we own $i = 2$; The value of i in is $\mu = 2M = 2^{J+1}$. The collocation points $x_l = \frac{l-0.5}{2M}$, with l =1,2, ...,$2M$ and the Haar function $h_i(x)$. The Haar coefficient matrix is known as $HAAR(i,l) = h_i(x_l)$
which is a square matrix $2M \times 2M$.

Method of Solution

We can assume that $H \epsilon L^2[0,1)$ is a Hilbert space with the inner product defined by
$\langle f(x), h \rangle = \int_0^1 f(x) h^T(x) dx$ Since V is a finite-dimensional subspace of$H$, it is closed and convex. Thus,
for (5) w $H,$ there are a unique approximation out of V such as g
$\|w - g\| \leq \|g - w\|, h\; V.$

$$f(x) = \sum_{i=0}^{2M} c_i \, h_i(x), \tag{6}$$

Where the HWM coefficients are given by

$c_i = \frac{(f(x),\; h_{(2M)}(x))}{(h_{(2M)}(x),\; h_{(2M)}(x))}$ Such that $(.,.)$ denotes the inner product.

Where $= 2M = 2^{J+1}$, C and $h(x)$ are $1 \times 2M$ matrices given by:

$$C = [c_1, c_2, c_3, \dots \dots, c_{2M}], \; h_{(2M)}(x) = [h_1(x) \;\; h_2(x) \;\; \dots \;\; h_{2M}(x)] \tag{7}$$

We can also approximate the function $k(x,t)\; L^2[0,1) \times [0,1)$ as follows:

$$k(x,t) \cong h^T{}_{(2M)}(x) \; K \; h_{(2M)}(x) \tag{8}$$

Where K is a $2M \times 2M$ matrix that we can acquire?

$$K = \frac{(h_{(2M)}(x),(k(x,t), h_{(2M)}(t))}{(h_{(2M)}(x), (h_{(2M)}(x))\ (h_{(2M)}(t), h_{(2M)}(t))} \tag{9}$$

Integration of Matrix

The integration of vector $h_{(\mu)}(\tau)$ is given by

$$\int_0^x h_{(\mu)}(\tau)\, d\tau \approx P_{(\mu\times\mu)}\, h_{(\mu)}(x)$$
(10)

Where $P_{(\mu\times\mu)}$is the $2M \times 2M$ operational matrix for integration Doing these integrations with the helpful of (10), we find;

$$P_{i,1}(x) = \begin{cases} x - \varsigma, & x\epsilon[\varsigma, \beta) \\ \gamma - x, & x\epsilon[\beta, \gamma) \\ o & elsewhere \end{cases} \tag{11}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \varsigma)^2, & x\epsilon\left[\varsigma, \beta\right), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2, & x\epsilon[\beta, \gamma), \\ \frac{1}{4m^2}, & x\epsilon[\gamma, 1), \\ 0, & elsewhere. \end{cases} \tag{12}$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x - \varsigma)^3, & x\epsilon\left[\varsigma, \beta\right), \\ \frac{1}{4m^2}(x - \beta) - \frac{1}{6}(\gamma - x)^3, & x\epsilon[\beta, \gamma), \\ \frac{1}{4m^2}(x - \beta), & x\epsilon[\gamma, 1), \\ 0, & elsewhere\ . \end{cases} \tag{13}$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x - \varsigma)^4, & x\epsilon\left[\varsigma, \beta\right), \\ \frac{1}{8m^2}(x - \beta)^2 - \frac{1}{24}(\gamma - x)^4 + \frac{1}{192m^4}, & x\epsilon[\beta, \gamma), \\ \frac{1}{8m^2}(x - \beta)^2 + \frac{1}{192m^4}, & x\epsilon[\gamma, 1), \\ 0, & elsewhere. \end{cases} \tag{14}$$

The Heaviside step function $g(x)$ is characterized as:

$$g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

By utilizing Heaviside step function, we can likewise write:

$$h_n(x) = g\left(x - \frac{k}{2^j}\right) - 2g\left(x - \frac{k+0.5}{2^j}\right) + g\left(x - \frac{k+1}{2^j}\right), \quad (15)$$

$n = 2^j + k, j, k \in N \cup \{0\}, \qquad 0 \le k \le 2^j$

The Integration of the Cross Product

The integration of the cross product of two Haar wavelets vectors can be obtained as:

$$D = \int_0^1 h_{(i)}(\tau)\, h^T{}_{(l)}(\tau) d(\tau) = \begin{cases} \frac{1}{m}, & i = l \\ 0, & i \neq l. \end{cases} \quad (16)$$

We discredited the functions $h_{(i)}(x)$ by dividing the interval $x \in [0,1]$ into $2M$ parts of equal length$\Delta x = \frac{1}{2M}$ and introduce the collocation points:

$$x_i = \frac{i - \frac{1}{2}}{2M}, \quad i = 1,2,\dots,2M.$$

Multiplication of HWM

It is important to assess the product of $h_{(\mu)}(x)$ and $h^T{}_{(\mu)}(x)$, that is called the product matrix of Haar wavelet method. Let
$M(x) \cong h_{(\mu)}(x)\, h^T{}_{(\mu)}(x)$
Where $M(x)$ is $2M \times 2M$matrix. Multiplying the matrix $M(x)$ by vector C we obtain
$M(x)C = \tilde{C} h_{(\mu)}(x)$ (17)

where $\tilde{C}$ is $2M \times 2M$ matrix and called the coefficient matrix.
Let R is $2M \times 2M$matrix. Multiplying the matrix R by vector $h_{(\mu)}(x)$
And multiplying the matrix $h_{(\mu)}(x)$ by the resulted matrix $R\ h_{(\mu)}(x)$ we obtain

$$h^T{}_{(\mu)}(x)\ R\ h_{(\mu)}(x) = \tilde{R} h_{(\mu)}(x) \quad (18)$$

where $\tilde{R}$ is $1 \times 2M$ matrix and called the coefficient matrix with the powerful properties of Eq. (17) We can achieve $\tilde{R}$ by away like $\tilde{C}$ we can convert the Volterra part of integral and Integro-Differential Equations System equations to an algebraic equation.

Error Analysis

Assume that $u(x)$ is the explicit solution and $u_{approx}(x)$ is the approximate solution of FVIDEs.
$ERROR_{approx}(x)$ Be the comparing error function and is defined as:

$$ERROR_{approx}(x) = \left|u(x) - u_{approx}(x)\right| \cong \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k}\, h_{2^j+k}\,(x).$$

Let $u(x)$ achieve a Lipschitz's condition on $[0,1]$, then there are constant $K > 0$ (subject to both $u(x)$ and interval), such that

$|u(x1) - u(x2)| \leq K|x1 - x2|, \forall x1, x2 \in [0,1]$.

The HWM will be convergent when $ERROR_{approx}(x)$ goes to zero as m goes to infinity. The request of convergence achieves as;

$$\left\|ERROR_{approx}(x)\right\|_2 = 0(\frac{1}{w})$$

Verification: Squaring the integrand and breaking the summation,

$$\left\|ERROR_{approx}(x)\right\|_2^{\,2} = \int_0^1 (\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k}\, h_{2^j+k}\,(x))^2\, dx$$

We can obtain:

$$\left\|ERROR_{approx}(x)\right\|_2^{\,2} = \sum_{j=J+1} \sum_{k=0}^{2^j-1} c^2{}_{2^j+k} \int_0^1 h^2{}_{2^j+k}(x)dx + \sum_{j=J+1} \sum_{k=0}^{2^j-1} \sum_{p=J+1} \sum_{q=0, q\neq k}^{2^p-1} c_{2^j+k}\, c_{2^p+q} \int_0^1 c_{2^j+k} c_{2^p+q}(x)dx$$

Using orthogonality conditions, we get:

$$\left\|ERROR_{approx}(x)\right\|_2^{\,2} = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c^2{}_{2^j+k} \left(\frac{1}{2^j}\right).$$

For $i = 2^j + k$, we obtain

$$c_{2^j+k} = 2^j \int_0^1 u(x)\, h_{2^j+k}(x)dx$$

From mean value theorem, there exist two points described below as:

$x_{11}^{jk} \in \left(\frac{k}{2^j}, \frac{k+0.5}{2^j}\right)$, $\quad x_{22}^{jk} \in \left(\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right)$,

So

$$c_{2^j+k} = 2^j[(\left(\frac{k+0.5}{2^j} - \frac{k}{2^j}\right) y\left(x_{11}^{jk}\right) + \left(\frac{k+1}{2^j} - \frac{k+0.5}{2^j}\right) y(x_{22}^{jk})$$

Using the Lipschitz's condition, we will get;

$$c_{2^j+k} = \frac{1}{2}[u(x_{11}^{jk}) - u(x_{22}^{jk})] \leq \frac{1}{2}k(x_{11}^{jk} - x_{22}^{jk})] \leq (\frac{1}{2})k\left(\frac{1}{2^j}\right) \cong k(\frac{1}{2^{j+1}})$$

$$\left\|ERROR_{approx}(x)\right\|_2{}^2 = \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} k^2 \frac{1}{2^{2J+2}}(\frac{1}{2^{2J}}) = \frac{k^2}{4}\sum_{j=J+1}^{\infty} 2^J\left(\frac{1}{2^{3J}}\right).$$

So
We will get:

$$\left\|ERROR_{approx}(x)\right\|_2{}^2 = \frac{k^2}{3}(\frac{1}{2^J+1})^2$$

But, $w = 2^J + 1$, so, we obtain,
$\left\|ERROR_{approx}(x)\right\|_2 = 0(\frac{1}{w})$ When w goes to infinity, then $ERROR_{approx}(x)$ tends to zero.

Description of the Proposed Approximation Technique

In this section, we introduce the presented method to solve the Volterra integro-differential equations in Esq. (1):
The parametric form of the denoted fuzzy integral equations system can be written as follows:

$$\underline{u}'(x,\alpha) = \underline{f}(x,\alpha) + \lambda\int_0^x k(x,t)\,\underline{u}\,(t,\alpha)dt \tag{19}$$

$$\overline{u}'(x,\alpha) = \overline{f}(x,\alpha) + \lambda\int_0^x k(x,t)\overline{u}\,(t,\alpha)dt \tag{20}$$

Where,

$$k(x,t)\underline{y}(t,\alpha) = \begin{cases} k(x,t)\,\underline{u}(t,\alpha), & k(x,t) \geq 0 \\ k(x,t)\,\overline{u}(t,\alpha), & k(x,t) < 0 \end{cases}$$

And,

$$k(x,t)\,\overline{u}(t,\alpha) = \begin{cases} k(x,t)\,\overline{u}\,(t,\alpha), & k(x,t) \geq 0 \\ k(x,t)\,\underline{u}(t,\alpha), & k(x,t) < 0 \end{cases} \tag{21}$$

We can approximate the function $\underline{u}(x,\alpha), \bar{u}(x,\alpha), \underline{f}(x,\alpha), \bar{f}(x,\alpha)$ and $k(x,t)$ by HWM as follows:

$$\underline{u}(x,\alpha) \approx h^T{}_{(\mu)}(x)\,\underline{U_1}\,h_{(\mu)}(\alpha) \quad , \quad \bar{u}(x,\alpha) \approx h^T{}_{(\mu)}(x)\,\overline{U_2}\,h_{(\mu)}(\alpha)$$

$$\underline{f}(x,\alpha) \approx h^T{}_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(\alpha)\ , \quad \bar{f}(x,\alpha) \approx h^T{}_{(\mu)}(x)\,\overline{F_2}h_{(\mu)}(\alpha) \tag{22}$$

$$\underline{u}'(x,\alpha) \approx h_{(\mu)}(\alpha)\,\underline{U_1}'^T h_{(\mu)}(x) \quad , \quad \bar{u}'(x,\alpha) \approx h_{(\mu)}(\alpha)\,\overline{U_2}'^T h_{(\mu)}(x)$$

$$k(x,t) = h^T{}_{(\mu)}(x)\,K\,h_{(\mu)}(t)$$

Which $\underline{u}'(x,\alpha)$ and $\bar{u}'(x,\alpha)$will be evaluated in terms $\underline{u}(x,\alpha)$ and $\bar{u}(x,\alpha)$

$$\underline{u}(x,\alpha) = \int_0^x \underline{u}'(t,\alpha)\,dt + \underline{u}(0)$$

If we expand $\underline{u}(0)$ with HWM basis i.e. $\underline{u}(0) = \underline{U_0}\, h_{(\mu)}(x)$ then U_0 is obtained as:

$$\underline{U_0} = \underbrace{\overbrace{\underline{u}(0),\underline{u}(0),\ldots\ldots,\underline{u}(0)},\ldots\ldots,\overbrace{\underline{u}(0),\underline{u}(0),\ldots\ldots,\underline{u}(0)}}_{2M} \tag{23}$$

$$\begin{aligned}\underline{U_1}^T h_{(\mu)}(x) &\cong \int_0^x \underline{U_1}^T h_{(\mu)}(t)\,dt + \underline{U_0}^T h_{(\mu)}(x)\\ &\cong \underline{U_1}^T \int_0^x h_{(\mu)}(t)\,dt + \underline{U_0}^T h_{(\mu)}(x)\\ &\cong \underline{U_1}^T P_{(\mu\times\mu)}\, h_{(\mu)}(x) + \underline{U_0}^T h_{(\mu)}(x)\\ &\cong (\underline{U_1}^T P_{(\mu\times\mu)} + \underline{U_0}^T)h_{(\mu)}(x)\end{aligned}$$

And we have

$$\underline{U_1}^T \cong (\underline{U_1}^T P_{(\mu\times\mu)} + \underline{U_0}^T \tag{24}$$

Therefore,

$$\underline{U_1}' \cong \left(P^T{}_{(\mu\times\mu)}\right)^{-1}(\underline{U_1} - \underline{U_0}) \tag{25}$$

After substituting the equations (22-28) into equations (20) and (21), we obtain;

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}\left(\underline{U_1} - \underline{U_0}\right) h_{(\mu)}(\alpha) = h^T{}_{(\mu)}(x)\,\underline{F_1}\, h_{(\mu)}(\alpha) + \lambda \int_0^x h^T{}_{(\mu)}(x)\, K\, h_{(\mu)}(t)\, h^T{}_{(\mu)}(t)\,\underline{U_1}\, h_{(\mu)}(\alpha)\,dt$$

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}(\overline{U_2} - \overline{U_0})h_{(\mu)}(\alpha) = h^T{}_{(\mu)}(x)\,\overline{F_2}\, h_{(\mu)}(\alpha) + \lambda \int_0^x h^T{}_{(\mu)}(x)\, K\, h_{(\mu)}(t)\, h^T{}_{(\mu)}(t) U_2\, h_{(\mu)}(\alpha)\,dt$$

(26)

We have:

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}\left(\underline{U_1} - \underline{U_0}\right) h_{(\mu)}(\alpha) = h^T{}_{(\mu)}(x)\,\underline{F_1}\, h_{(\mu)}(\alpha) + \lambda\, h^T{}_{(\mu)}(x)\, K \int_0^x h_{(\mu)}(t)\, h^T{}_{(\mu)}(t)\,dt\ \underline{U_1}\, h_{(\mu)}(\alpha)$$

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}(\overline{U_2} - \overline{U_0})h_{(\mu)}(\alpha) = h^T{}_{(\mu)}(x)\,\overline{F_2}\, h_{(\mu)}(\alpha) + \lambda\, h^T{}_{(\mu)}(x)\, K \int_0^x h_{(\mu)}(t)\, h^T{}_{(\mu)}(t)\,dt U_2\, h_{(\mu)}(\alpha)$$

With the effective properties of equation (10) we get:

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}\left(\underline{U_1}-\underline{U_0}\right)h_{(\mu)}(\alpha)$$
$$= h^T{}_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(\alpha)\ +\lambda\,h^T{}_{(\mu)}(x)\,K\,\underline{\widetilde{U_1}}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha)$$

$$h^T{}_{(\mu)}(x)\left(P^T{}_{(\mu\times\mu)}\right)^{-1}(\overline{U_2}-\overline{U_0})\,h_{(\mu)}(\alpha) =$$
$$h^T{}_{(\mu)}(x)\,\overline{F_2}\,h_{(\mu)}(\alpha) + \lambda\,h^T{}_{(\mu)}(x)\,K\,\overline{\breve{U}_2}\;P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha)$$
(27)

Therefore

$$\left(P^T{}_{(\mu\times\mu)}\right)^{-1}\left(\underline{U_1}-\underline{U_0}\right) = \underline{F_1}\,h_{(\mu)}(\alpha)\ +\lambda\,h^T{}_{(\mu)}(x)\,K\,\underline{\widetilde{U_1}}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha) \quad (28)$$

$$\left(P^T{}_{(\mu\times\mu)}\right)^{-1}(\overline{U_2}-\overline{U_0})\,h_{(\mu)}(\alpha) =$$
$$\overline{F_2}\,h_{(\mu)}(\alpha) + h^T{}_{(\mu)}(x)\,K\,\overline{\breve{U}_2}\;P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha)$$
(29)

Where, the dimensional describes are falling to define the notation. Rewriting (29), we have

$$\underline{U_1} = P^T{}_{(\mu\times\mu)}(\underline{F_1}\,h_{(\mu)}(\alpha)\ +\lambda\,h^T{}_{(\mu)}(x)\,K\,\underline{\widetilde{U_1}}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha)+\underline{U_0}$$
$$\overline{U_2} = P^T{}_{(\mu\times\mu)}(\overline{F_2}\,h_{(\mu)}(\alpha)+\ \lambda\,h^T{}_{(\mu)}(x)\,K\,\overline{\breve{U}_2}\;P_{(\mu\times\mu)}\,h_{(\mu)}(x)\,h_{(\mu)}(\alpha)+\overline{U_0} \quad (30)$$

From Esq. (30) we have a system of $2M\times 2M$ nonlinear equations and $2M$ unknowns. After solving above nonlinear system using Newton method. We can achieve the unknown vectors $\underline{U_1}$ and $\overline{U_2}$. The required approximated solutions. $\underline{u}(x,\alpha)\approx h^T{}_{(\mu)}(x)\,\underline{U_1}\,h_{(\mu)}(\alpha)$, $\overline{u}(x,\alpha)\approx h^T{}_{(\mu)}(x)\,\overline{U_2}\,h_{(\mu)}(\alpha)$ for the FVIDE.

Numerical Examples

Two examples to illustrate the HWM functions for nonlinear fuzzy Fredholm and Volterra integrodifferential equations are considered. In this case, fuzzy approximate solutions using the HWM functions are given in Table1 the HWM functions is less absolute error than homotopy perturbation method [14]. In Table 1 the HWM functions method and homotopy perturbation method have been compared. The HWM functions method and the exact solutions have been compared in Table 2. The HWM has less error compared with homotopy perturbation method.

Example 14. Consider the following nonlinear Fuzzy Fredholm Integro-differential Equations of the second kind [31] with

$$y'(x,r)=(r-\frac{r^2}{8},\frac{12-4r-r^2}{8})+\int_0^1 \frac{t^2}{2}\,y^2(t)\,dt\ , \qquad 0\le x,t<1 \text{ and } \lambda=1 \tag{31}$$

Here

$$\underline{f}(x,r)=r-\frac{r^2}{8}\ ,\ \bar{f}(x,r)=\frac{12-4r-r^2}{8}\ \ ,\ \underline{k}(x,t)=\overline{k(x,t)}=\frac{t^2}{2}$$

The exact solution is given by
$\underline{y}(x,r)=r\,x,\ \bar{y}(x,r)=(2-r)x,$

By using Maple program (Maple Package Version 17), these equations are solved to get the components of the above iterations. Fuzzy approximate solutions are given at $2M=8$ and are given in Table 1 which illustrates the obtained approximate solution compared to the exact solution subject to the initial conditions. The values for *x*□ *0.5* are found and it is noticeable that the approximate solutions are near the exact solutions.

Table 1
Comparison between the exact, HWM and HPM results of y(x,r), $\overline{y}$(x,r) for Example 14

	Exact solution		**Approximate solution** (proposed method)		**Error of HWM**		**Error of HPM**	
r	$\underline{y}(x,r)$	$\overline{y}(x,r)$	$\underline{y}(x,r)$	$\overline{y}(x,r)$	$\underline{y}(x,r)$	$\overline{y}(x,r)$	$\underline{E}(x,r)$	$\overline{E}(x,r)$
0.1	0.05	0.95	0.0498979	0.949531	0.0001021	0.000469	0.001289	0.006342
0.2	0.1	0.9	0.0995769	0.899726	0.0004231	0.000274	0.002753	0.003765
0.3	0.15	0.85	0.1489982	0.849739	0.0010018	0.000261	0.003893	0.003680
0.4	0.2	0.8	0.199273	0.799642	0.000727	0.000358	0.002476	0.007675
0.5	0.25	0.75	0.249790	0.749537	0.000210	0.000463	0.000880	0.000144
0.6	0.3	0.7	0.299894	0.699853	0.000106	0.000147	0.000229	0.000238
0.7	0.35	0.65	0.3499146	0.649859	0.0000854	0.000141	0.000374	0.000177
0.8	0.4	0.6	0.3999531	0.599816	0.0000469	0.000184	0.000288	0.000265
0.9	0.45	0.55	0.4499862	0.549867	0.0000138	0.000133	0.000748	0.000247

Example 15. Consider the nonlinear fuzzy Fredholm integro-differential equations of the second kind [32] with

$$\underline{y}'(x,r)=(r-\frac{x^2\,r^2}{40}),$$

$$\bar{y}'(x,r)=(2-r-\frac{x^2(2-r)^2}{40}), \tag{32}$$

And

$$k(x,t)=\frac{x^2 t}{10},\quad 0\le x,t<1 \text{ And } \lambda=1.$$

The exact solution is given by

$$\underline{y}(x,r)=r\,x,\ \ \bar{y}(x,r)=(2-r)x,$$

Table 2 demonstrates the numerical results for this example and compares the maximum absolute errors of HWM with the exact solution for different value of M. These results have been included to demonstrate the validity and capability of HWM and the collocation method based on the familiar Newton method.

Table 2
Comparison of Absolute error for HWM method at x = 0.5 of Example 15.

	Exact solution		Approximate solution at $2M=8$		Error of HWM at $2M=8$		Error of HWM at $2M=16,$	
r	$\underline{y}(x,r)$	$\overline{y}(x,r)$	$\underline{y}(x,r)$	$\overline{y}(x,r)$	$\underline{E}(x,r)$	$\overline{E}(x,r)$	$\underline{E}(x,r)$	$\overline{E}(x,r)$
0.1	0.05	0.95	0.0500000289	0.9500106732	2.897×10^{-8}	1.06732×10^{-6}	7×10^{-14}	5.3×10^{-12}
0.2	0.10	0.90	0.1000001160	0.9000095680	1.160×10^{-7}	9.5680×10^{-6}	9×10^{-13}	1.9×10^{-12}
0.3	0.15	0.85	0.1500002613	0.8500085250	2.613×10^{-7}	8.5250×10^{-6}	4×10^{-13}	7.2×10^{-12}
0.4	0.20	0.80	0.2000003679	0.8000075427	3.679×10^{-7}	7.5427×10^{-6}	6×10^{-13}	9.1×10^{-12}
0.5	0.25	0.75	0.2500007275	0.7500066218	7.275×10^{-7}	6.6218×10^{-6}	2×10^{-13}	8.5×10^{-12}
0.6	0.30	0.70	0.3000000614	0.7000057617	6.15×10^{-8}	5.7617×10^{-6}	5×10^{-13}	5.3×10^{-12}
0.7	0.35	0.65	0.3500007156		7.156×10^{-7}	4.9624×10^{-6}	8×10^{-13}	4.7×10^{-12}
0.8	0.40	0.60	0.4000087645	0.6000042235	8.7645×10^{-6}	4.2235×10^{-6}	1.6×10^{-12}	9.3×10^{-12}
0.9	0.45	0.55	0.4500023677	0.5500035449	2.367×10^{-6}	8.7645×10^{-6}	7.3×10^{-12}	8.6×10^{-12}

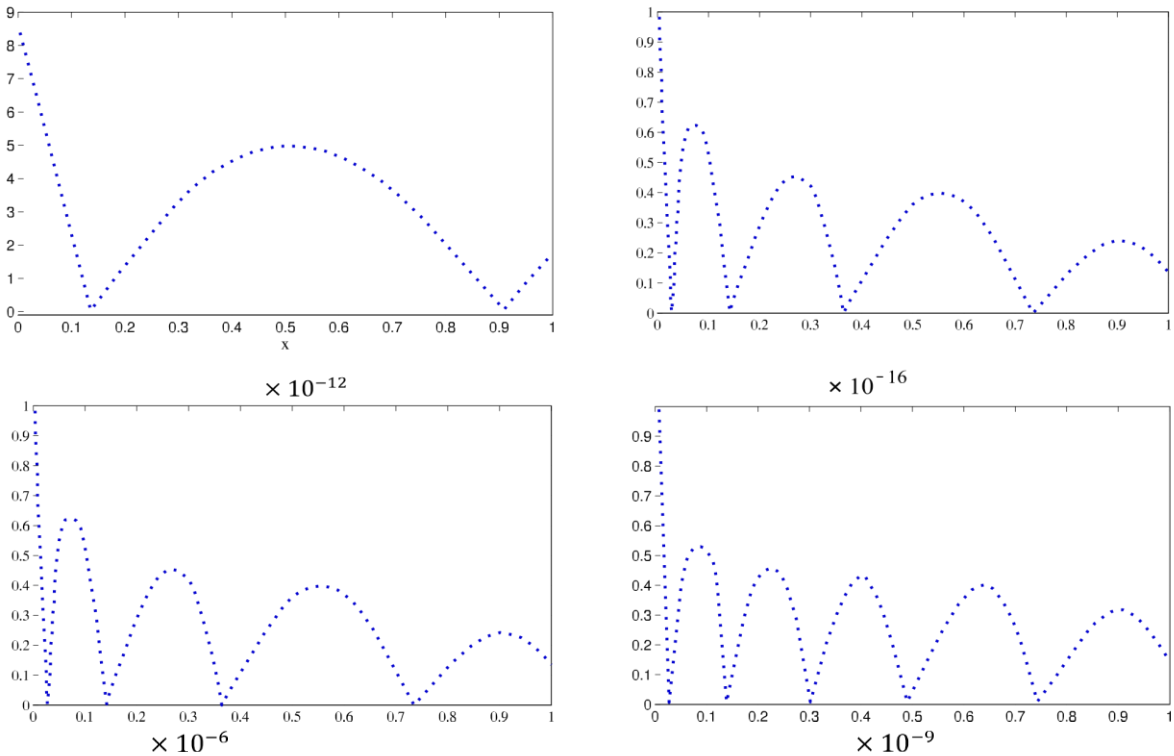

Figure 1. Plot of the absolute error $y(x,\alpha)$ at M = 4,8,12 and 16 for Example 2

Conclusion

We used an appropriate technique for finding the approximate solution of FVIDE. We conclude here from the results that the HWM is a promising tool for this type of fuzzy integral equations. We calculate operational matrices and apply them for solving FVIDE. By applying these matrices and collocation method FVIDE convert to linear systems of algebraic equations which can be solved by a suitable approximation method and needing small CPU time to provide better solutions.

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