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Investigation of space-fractional diffusion equations via PGS iterative method

Howard Crane
Wayne State University, Michigan, United States

***Abstract*---**To speed up the convergence rate in solving the linear system iteratively, we construct the corresponding preconditioned linear system. Then we formulate and implement the Preconditioned GaussSeidel (PGS) iterative method for solving the generated linear system. One example of the problem is presented to illustrate the effectiveness of the PGS method. The numerical results of this study show that the proposed iterative method is superior to the basic GS iterative method. In this paper, we deal with the application of an unconditionally implicit finite difference approximation equation of the one-dimensional linear space-fractional diffusion equations via the Caputo's space-fractional derivative. Based on this implicit approximation equation, the corresponding linear system can be generated in which its coefficient matrix is large scale and sparse.

***Keywords*---**Matrix, Implicit, PGS method, GS iterative method, linear.

Introduction

Based on previous studies in [1, 2, 3, and 4] many successful mathematical models, which are based on fractional partial derivative equations (FPDEs), have been developed. Following that, there are several methods used to solve these models. For instance, we have a transform method [5], which is used to obtain analytical and/or numerical solutions of the fractional diffusion equations (FDE). Other than this method, other researchers have proposed finite difference methods such as explicit and implicit [6, 7, and 8]. Also, it is pointed out that the explicit methods are conditionally stable. Therefore, we discredited the spacefractional diffusion equation via the implicit finite difference discretization scheme and Caputo's fractional partial derivative of order in order to derive a Caputo's implicit finite difference approximation equation. This approximation equation leads to a tridiagonal linear system.

Due to the properties of the coefficient matrix of the linear system which is a sparse and large scale, iterative methods are the alternative option for efficient

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solutions. As far as iterative methods are concerned, it can be observed that many researchers such as Ghuang-hui [9], Young [10], Hackbusch [11], and Saad [12] have proposed and discussed several families of iterative methods. In addition to that, the concept of block iteration has also been introduced by Evans [13], Ibrahim and Abdullah [14], Evans and Yousif [15] to demonstrate the efficiency of its computation cost. Among the existing iterative methods, the preconditioned iterative methods (Ghuang-Hui [9], Zhao [16], Hoang-hao [17], Gunawardena [18], Saad [12]) have been widely accepted to be one of the efficient methods for solving linear systems. Because of the advantages of these iterative methods, the aim of this paper is to construct and investigate the effectiveness of the Preconditioned Gauss-Seidel (PGS) iterative method for solving space-fractional parabolic partial differential equations (SPPDE's) based on the Caputo's implicit finite difference approximation equation. To investigate the effectiveness of the PGS method, we also implement the Gauss-Seidel (GS) iterative methods being used as a control method. To demonstrate the effectiveness of the PGS method, let space-fractional parabolic partial differential equation (SPPDE's) be defined as

$$\frac{\partial U(x,t)}{\partial t} = a(x)\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) + f(x,t) \qquad \textbf{(1)}$$

With initial condition $U(x,0) = f(x),\ 0 \le x \le \lambda,$ and
Boundary conditions $U(0,t) = g_0(t),\ 0 < t \le T,\ U(\lambda,t) = g_1(t),\ 0 < t \le T.$

The outline of this paper is organized as follows: In Section 2 and 3, an approximate the formula of the Caputo's fractional derivative operator and numerical procedure for solving space- fractional diffusion equation (1) by means of the implicit finite difference method is given. In Section 4, the formulation of the PGS iterative method is introduced. Section 5 shows numerical example and its results and conclusion is given in section 6.

Preliminaries

Before constructing the linear systems, some definitions that can be applied for fractional derivative theory need to develop the approximation equation of problem (1) in

Definition 1 [5] The Riemann-Liouville fractional integral operator, J of order- is defined as

$$J^{\beta} f(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1} f(t)dt,\ \beta > 0,\ x > 0 \qquad (2)$$

Definition 2 [1] The Caputo's fractional partial derivative operator, D^{β} of order - β is defined as

$$D^{\beta}f(x)=\frac{1}{\Gamma(m-\beta)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta-m+1}}dt,\ \beta>0 \qquad (3)$$

With

$$m-1<\beta\le m,\ m\in N,\ x>0.$$

We have the following properties when

$$m-1<\beta\le m,\ x>0:$$

$D^{\beta}k=0$, (k is a constant),

$$D^{\beta}x^{n}=\begin{cases}0, & \text{for } n\in N_0 \text{ and } n<[\beta]\\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\beta)}x^{n-\beta}, & \text{for } n\in N_0 \text{ and } n\ge[\beta]\end{cases}$$

Where function $[\beta]$ denotes the smallest integer greater than or equal to β, $N_0=\{0,1,2,\ldots\}$ and $\Gamma(.)$ is the gamma function.

Derivation of Caputo's Implicit Finite Difference Approximation

Assume that $h=\frac{\lambda}{k}$, k is positive integer and using second order approximation, we get

$$\frac{\partial^{\beta}U(x_i,t_n)}{\partial x^{\beta}}=\frac{1}{\Gamma(2-\beta)}\int_0^{t_n}\frac{\partial^2 U(x_i,s)}{\partial x^2}(t_n-s)^{1-\beta}\partial s$$

$$=\frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\int_{jh}^{(j+1)h}\left(\frac{U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}}{h^2}\right)(nh-s)^{\beta}\partial s=\frac{h^{-\beta}}{\Gamma(3-\beta)}\sum_{j=0}^{i-1}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)\left((j+1)^{2-\beta}-j^{2-\beta}\right) \qquad (4)$$

Let us define

$$\sigma_{\beta,h}=\frac{h^{-\beta}}{\Gamma(3-\beta)}$$

And

$$g_j^{\beta}=(j+1)^{2-\beta}-j^{2-\beta}$$

Then the discrete approximation of Eq. (4)

$$\frac{\partial^{\beta}U(x_i,t_n)}{\partial x^{\beta}}=\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)$$

Now we approximate Problem (1) by using Caputo's implicit finite difference approximation:

$$\lambda\left(U_{i,n}-U_{i,n-1}\right)=a_i\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)+b_i\frac{\left(U_{i+1,n}-U_{i-1,n}\right)}{2h}+C_iU_{i,n}+f_{i,n}$$

For *I=1, 2… m-1.* Then we can simplify the scheme approximation equation as

$$\lambda U_{i,n-1}=-a_i\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)\qquad-\frac{b_i}{2h}\left(U_{i+1,n}-U_{i-1,n}\right)-C_iU_{i,n}+\lambda U_{i,n}-f_{i,n}$$

So we get:

$$\therefore b_i^*U_{i-1,n}+\left(\lambda-c_i^*\right)U_{i,n}-b_i^*U_{i+1,n}$$
$$-a_i^*\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)=f_i \qquad (5)$$

Where

$$a_i^*=a_i\sigma_{\beta,h}\ ,\ \ b_i^*=\frac{b_i}{2h}\ ,\ \ c_i^*=c_i,\ F_i^*=f_{i,n}$$

And

$$f_i=\lambda\left(U_{i,n-1}\right)+F_i^*$$

According to Eq. (5), the approximation equation is known as the fully implicit finite difference approximation equation which is consistent second order accuracy in space-fractional. For simplicity, let Eq. (5) for $n>3$ be rewritten as

$$-R_i+\alpha_iU_{i-3,n}+s_iU_{i-2}+p_iU_{i-1,n}+q_iU_{i,n}+r_iU_{i+1,n}=f_i \qquad (6)$$

Where

$$R_i=a_i^*\sum_{j=3}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right),$$
$$\alpha_i=\left(-a_i^*g_2^{\beta}\right),$$
$$s_i=\left(-a_i^*g_1^{\beta}+2a_i^*g_2^{\beta}\right),$$
$$p_i=\left(b_i^*-a_i^*g_2^{\beta}+2a_i^*g_1^{\beta}-a_i^*\right),$$
$$q_i=\left(-a_i^*g_1^{\beta}+2a_i^*+\left(\lambda-c_i^*\right)\right),$$
$$r_i=\left(-a_i^*-b_i^*\right)$$

Then Eq. (6) can be used to construct a linear system in matrix form as

$$A\underset{\sim}{U}=\underset{\sim}{f} \qquad (7)$$

Where

$$A=\begin{bmatrix} q_1 & r_1 & & & & & & & \\ p_2 & q_2 & r_2 & & & & & & \\ s_3 & p_3 & q_3 & r_3 & & & & & \\ \alpha_4 & s_4 & p_4 & q_4 & r_4 & & & & \\ & \alpha_5 & s_5 & p_5 & q_5 & r_5 & & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & \alpha_{m-2} & s_{m-2} & p_{m-2} & q_{m-2} & r_{m-2} \\ & & & & \alpha_{m-1} & s_{m-1} & p_{m-1} & q_{m-1} \end{bmatrix}_{(m-1)x(m-1)}$$

$$\underset{-}{U}=\begin{bmatrix} U_{1,1} & U_{2,1} & U_{3,1} & \Lambda & U_{m-2,1} & U_{m-1,1} \end{bmatrix}^T$$

$$\underset{-}{f}=\lfloor f_1-p_1U_{1,1} \quad f_2+s_2U_{2,1} \quad f_3+\alpha U_{3,1} \quad f_4+R_4 \ \ldots$$

$$f_{m-2,1}+R_{m-2} \quad f_{m-1,1}-p_{m-1}U_{m,1}+R_{m-1}\rfloor^T$$

Formulation of Preconditioned Gauss-Seidel Iterative Method

In relation to the tridiagonal linear system in Eq. (7), it is clear that the characteristics of its coefficient matrix are large scale and sparse. As mentioned in Section 1, many researchers have discussed various iterative methods such as Ghuang-Hui [9], Zhao [16], Hoang-hao [17], Gunawardena [18], Young [10], Hackbusch [11], Saad [12], Yousif and Evans [15]. To obtain numerical solutions of the tridiagonal linear system (8), we consider the Preconditioned Gauss-Seidel (PGS) iterative method [9, 16, 17, 18], which is the most known and widely used for solving any linear systems. Before applying the PGS iterative method, we need to transform the original linear system (7) into the preconditioned linear system

$$A^* \underset{\sim}{x} = \underset{\sim}{f}^* \qquad (8)$$

Where,

$$A^* = PAP^T,$$

$$\underset{\sim}{f}^* = P\underset{\sim}{f}, \quad \underset{-}{U} = P^T\underset{-}{x}.$$

Actually, the matrix P is called a preconditioned matrix and defined as [19]

$$P = I + S$$

Where

$$S=\begin{bmatrix} 0 & -r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_3 & 0 & 0 \\ 0 & 0 & O & O & O & 0 \\ 0 & 0 & 0 & 0 & 0 & -r_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)x(m-1)}$$

and the matrix I is an identical matrix. To formulate PGS method, let the coefficient matrix A^* in (7) be expressed as summation of the three matrices

$$A^* = D - L - V \qquad (9)$$

Where
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D, L and V are diagonal, lower triangular and upper triangular matrices respectively. By using Eq. (9) and (11), the formulation of PGS iterative method can be defined generally as [9, 16, 17, 18, and 19]

$$\underset{\sim}{x}^{(k+1)} = (D-L)^{-1} V \underset{\sim}{x}^{(k)} + (D-L)^{-1} \underset{\sim}{f}^{*} \qquad (10)$$

Where $\underset{\sim}{x}^{(k+1)}$ represents an unknown vector at *(k+1)* the iteration.

Numerical Example

In this section, we have examples of the space-fractional diffusion equations to verify the effectiveness of the Gauss-Seidel (GS) and Preconditioned Gauss-Seidel (PGS) iterative methods. In comparison, three criteria will be considered for iterative methods such as number iterations, the execution time (seconds), and maximum error at three different values of $\beta = 1.2$, $\beta = 1.5$ and $\beta = 1.8$. During the implementation of the point iterations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$.
Let us consider the following space-fractional initial boundary value problem

$$\frac{\partial U(x,t)}{\partial t} = d(x)\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} + p(x,t), \qquad (11)$$

On finite domain $0 < x < 1$, with the diffusion coefficient $d(x) = \Gamma(\beta) x^{0.5}$, the source function $p(x,t) = (x^2+1)\cos(t+1) - 2x\sin(t+1)$,
With the initial condition $U(x,0) = (x^2+1)\sin(1)$ and the boundary condition $U(0,t) = \sin(t+1)$, $U(1,t) = 2\sin(t+1)$, For t>0. The Exact solution of this problem is $U(x,t) = (x^2+1)\sin(t+1)$

All numerical results for problems (11), obtained from application of GS and PGS iterative methods are recorded in Tables 1 by using the different value of mesh size, M=128, 256, 512, 1024 and 2048.

Conclusion

In order to get the numerical solution of the space-fractional diffusion problems, the paper presents the derivation of the Caputo's implicit finite difference approximation equations in which this approximation equation leads to a linear system. From the observation of all experimental results by imposing the GS and PGS iterative methods, it is obvious at the number of iterations has declined approximately by 41.3082.45% corresponds to the PGS iterative method compared with the GS method. Again in terms of execution time, implementations of the PGS method are much faster about 51.18-92.43% than the GS method. It means that the PGS method requires the least amount for the number of iterations and computational time as compared with GS iterative methods. Based on the accuracy of both iterative methods, it can be concluded that their numerical solutions are in good agreement.

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